

# Random Surfaces in Four Dimensions and One-Dimensional String Theory

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## Abstract

We consider a new action of a two-dimensional field theory interacting with gravitational field. The action is interpreted as the area of a surface imbedded into four-dimensional Minkowski target space. In addition to reparametrization invariance the new action has one extra infinite-dimensional local symmetry with a clear geometrical meaning. The special gauge choice, which includes the gauge condition of tracelessness of the energy-momentum tensor, leads to an effective free scalar field theory. The problem of anomalies in quantum theory and possible connection with matrix quantum mechanics are also discussed.

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## 1. Introduction

In recent years a lot of work was done in the field of one-dimensional string theories, i.e. string theories with one dimensional target space (for a review see, for example [1]). These theories are connected with matrix models and integrable systems and that makes possible the calculation of the correlation functions in all orders of perturbation expansion over Riemann surfaces. However it is difficult to generalize this connection (and to apply the corresponding powerful techniques) for the most physically interesting case of  $D$ -dimensional string theories ( $D > 1$ ). Instead, the one-dimensional string theories one usually consider as a toy models.

In this work we are going to argue that some definite **one-dimensional string theory**, i.e. some field theory on two-dimensional manifold of one scalar field  $\phi(\sigma, \tau)$  interacting with (two-dimensional) metric  $g_{\mu\nu}(\sigma, \tau)$  effectively could be interpreted as a string theory in **four-dimensional** Minkowski space.

The idea is, roughly speaking, to consider the two-dimensional metric  $g_{\mu\nu}$  which appears in the theory as the metric, induced by an imbedding of the two-dimensional world-sheet into **three-dimensional** Euclidean space, and to consider the only field  $\phi$  as the remaining time-coordinate  $X^0$ . In other words, one can encode the **three** functions  $X^i(\sigma, \tau)$  ( $i = 1, \dots, 3$ ) corresponding to Euclidean coordinates into new dynamical variables  $g_{\mu\nu}(\sigma, \tau)$  ( $\mu, \nu = 0, 1$ ) as follows:

$$g_{\mu\nu} = \partial_\mu X^i \partial_\nu X^i$$

Then it is easy to rewrite in new terms the area of the world-sheet thus obtaining an action depending on  $g_{\mu\nu}$  and  $\phi$ , and in quantization in the path integral one may replace the integration over imbeddings  $X^\alpha(\sigma, \tau)$  where  $\alpha = 0, \dots, 3$  with the integration over  $g_{\mu\nu}(\sigma, \tau)$  and  $\phi(\sigma, \tau)$ . When replacing, one should avoid possible missing or overcounting of imbedded surfaces. This is subtle point and it will be considered in Section 5. **Let us emphasize that this would be not a standard string theory.**

Let us also notice the case  $D = 4$  is special in such an approach, since in this case the number of Euclidean coordinates (three) coincides with the number of components of a two-dimensional metric. Because of that, in four-dimensional case the number of dynamical fields (over which goes integration in the path integral) remains unchanged. Also for that case under certain conditions (see Section 5) the  $X^i(\sigma, \tau)$  can be restored unambiguously from  $g_{\mu\nu}(\sigma, \tau)$  and, hence, there is a possibility to define **four-dimensional** correlational functions in the theory (see Section 5).

Although this approach could seem artificial, it could provide us with a possibility of the calculation of correlational functions of a four-dimensional string theory using the technology of matrix models and integrable theories. Actually, this is the ultimate goal of this consideration.

## 2. Action

Let us consider the following nonlinear functional of  $g^{\mu\nu}(z)$  ( $\mu, \nu = 1, 2$ ) and  $\phi(z)$  where  $z = (z^1, z^2)$  are coordinates on some two-dimensional manifold  $\Sigma_2$ :

$$S[g_{\mu\nu}, \phi] = \int d^2z \sqrt{g} \sqrt{g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 1} \quad (2.1)$$

as an action of two-dimensional field theory with gravitational field. If one want to consider only time-like surfaces in the framework of four-dimensional interpretation of (2.1) (see Section 5), than the domain of definition of the metric  $g_{\mu\nu}(z)$  and of the one scalar field  $\phi(z)$  should be defined by the condition of reality of the action  $S$ , i.e.

$$F \equiv g(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 1) > 0 \quad (2.2)$$

This condition is fully analogous to the usual condition of the reality of the Nambu-Goto action:

$$(\dot{X} X')^2 - \dot{X}^2 X'^2 > 0 \quad (2.3)$$

However, in the recent work of Carlini and Greensite [2] it was argued that in case of square-root actions (like (2.1) ) one should in path integral integrate over time-like **and** space-like trajectories (or surfaces) in order to obtain unitarity and finiteness of the theory. So, the question: should one impose the condition (2.2) , is still open (at least for the author).

Let us note that the action (2.1) can be rewritten as follows

$$S[g_{\mu\nu}, \phi] = \int d^2z \sqrt{\det(g_{\mu\nu} - \partial_\mu \phi \partial_\nu \phi)} \quad (2.4)$$

If one consider  $g_{\mu\nu}(z)$  as a dynamical variable than the theory, defined by the action (2.1) , is not a standard one in a sense that its equations of motion (for  $g^{\mu\nu}$  or  $g_{\mu\nu}$ ) are contradictory (if there is only *one* field  $\phi$ ). The same problem (absence of equations of motion for a metric or degeneracy of a metric) actually arises in **any** two-dimensional theory of **one** scalar field coupled to gravity. The reason is very simple: the equations of motion for the metric  $g^{\mu\nu}$  in such theories always look like:

$$A(z) \partial_\mu \phi \partial_\nu \phi + B(z) g_{\mu\nu} = 0$$

and, hence,

$$\det(g_{\mu\nu}) = \det\left(-\frac{A(z)}{B(z)} \partial_\mu \phi \partial_\nu \phi\right) \equiv 0$$

The familiar examples are Polyakov action [3] with one-dimensional target space, which eventually produces 2-dimensional critical string theory with additional Liouville field (see [4]):

$$S_{Polyakov} = \frac{1}{4\pi\alpha'} \int d^2z \sqrt{g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \Lambda)$$

and Liouville action [1]:

$$S_{Liouville} = \frac{1}{8\pi} \int d^2z \sqrt{g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + Q\phi R(g) + \frac{\mu}{\gamma^2} e^{\gamma\phi})$$

However if we consider  $g_{\mu\nu}$  as an external field and not a dynamical variable, the equation of motion for the dynamical field  $\phi(z)$  exists:

$$\Phi(g_{\mu\nu}(z), \phi(z)) \equiv \frac{1}{\sqrt{\det(g)}} \frac{\delta S[g_{\mu\nu}, \phi]}{\delta \phi(z)} = D^\rho \left( \frac{\partial_\rho \phi}{\sqrt{g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 1}} \right) = 0 \quad (2.5)$$

We can also calculate the energy-momentum tensor of the theory (2.1) using the standard definition:

$$\begin{aligned} T_{\mu\nu}(g_{\rho\sigma}(z), \phi(z)) &= \frac{1}{\sqrt{g}} \frac{\delta S[g_{\rho\sigma}, \phi]}{\delta g^{\mu\nu}(z)} \\ &= \frac{1/2}{\sqrt{g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi - 1}} (\partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} (g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi - 1)) \end{aligned} \quad (2.6)$$

It turns out that the action  $S$  has some intriguing properties, namely:

**1.** The action  $S$  besides the two-parametric reparametrization invariance has else one infinite-dimensional local symmetry. All these symmetries allow to fix a gauge (locally) so that the resulting theory becomes the theory of a free scalar massless field. That new symmetry has a clear geometrical meaning.

**2.** The action  $S$  equals to the area of the two-dimensional manifold  $\Sigma_2$  imbedded into four-dimensional Minkowski space whereupon this imbedding is given by  $g_{\mu\nu}$  and  $\phi$ . Hence, when constructing a quantum theory of random surfaces in four dimensions, one may try to replace the path integration over  $X^\alpha(z)$  ( $\alpha = 0, \dots, 3$ ) which usually describe an imbedding with the path integration over  $g_{\mu\nu}(z)$  and  $\phi(z)$  and, expressing  $X^\alpha(z)$  via  $g_{\mu\nu}(z)$  and  $\phi(z)$ , define four-dimensional correlational functions.

**3.** The theory defined by  $S$  has no anomalies, at least naively (i.e. the symmetries of the action are preserved in regularized theory).

### 3. Symmetries

The action  $S$  is evidently invariant under coordinate transformations :

$$S' \equiv S[g'_{\mu\nu}, \phi'] = S[g_{\mu\nu}, \phi]$$

where

$$\begin{aligned} z' &= z'(z) \\ \phi'(z') &= \phi(z) \\ g'_{\mu\nu}(z') &= \frac{\partial z^\rho}{\partial z'^\mu} \frac{\partial z^\sigma}{\partial z'^\nu} g_{\rho\sigma}(z) \end{aligned}$$

This invariance is equivalent to the following identity for the energy-momentum tensor introduced in Section 2:

$$D_\mu T^{\mu\nu} \equiv 0 \quad (3.1)$$

where  $D_\mu$  is the covariant derivative.

Furthermore, there is else one new infinite-dimensional symmetry of the action (which we will further call "form-symmetry"):

$$S^\zeta \equiv S[g^\zeta_{\mu\nu}, \phi^\zeta] = S[g_{\mu\nu}, \phi]$$

where the "form-transformations" are

$$g^\zeta_{\mu\nu}(z) = g_{\mu\nu}(z) + \partial_{(\mu} \phi \partial_{\nu)} \zeta + \partial_\mu \zeta \partial_\nu \phi \quad (3.2a)$$

$$\phi^\zeta(z) = \phi(z) + \zeta(z) \quad (3.2b)$$

where  $\zeta(z)$  is an *arbitrary* function of  $z$ .

The invariance of the action (2.1) with respect to form-transformations is equivalent to the following identity between the energy momentum tensor  $T^{\mu\nu}$  and the  $\Phi$  introduced in Section 2:

$$\Phi \equiv T^{\mu\nu} D_\mu \partial_\nu \phi \quad (3.3)$$

where identity (3.1) was taken into account.

One can also mention that the generalization of the action (2.4)

$$S[g_{\mu\nu}, \phi^i] = \int d^2 z \sqrt{\det(g_{\mu\nu} - \partial_\mu \phi_i \partial_\nu \phi_i)} \quad (3.4)$$

where  $\phi_i(z)$  ( $i = 1, \dots, D$ ) are scalar fields, is also form-symmetric if  $D > 1$ , i.e. symmetric under the following transformations:

$$g_{\mu\nu}^\zeta(z) = g_{\mu\nu}(z) + \partial_{(\mu}\phi_i\partial_{\nu)}\zeta_i + \partial_\mu\zeta_i\partial_\nu\zeta_i \quad (3.5a)$$

$$\phi_i^\zeta(z) = \phi_i(z) + \zeta_i(z) \quad (3.5b)$$

where  $\zeta_i(z)$  are arbitrary functions of  $z$ . The action (3.4) has no equations of motion for metric  $g_{\mu\nu}$  as well.

At the same time the possible generalization of the action in the form (2.1) :

$$S[g_{\mu\nu}, \phi^i] = \int d^2z \sqrt{g} \sqrt{g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i - 1} \quad (3.6)$$

where  $\phi^i(z)$  ( $i = 1, \dots, D$ ) are scalar fields, is **not** form-symmetric if  $D > 1$ .

Let us also notice that form-transformations are abelian and have the summation rule  $\zeta_{12} = \zeta_1 + \zeta_2$ , i.e.

$$\begin{aligned} (g_{\mu\nu}^{\zeta_1})^{\zeta_2} &= (g_{\mu\nu}^{\zeta_2})^{\zeta_1} = g_{\mu\nu}^{\zeta_1 + \zeta_2} \\ (\phi^{\zeta_1})^{\zeta_2} &= (\phi^{\zeta_2})^{\zeta_1} = \phi^{\zeta_1 + \zeta_2} \end{aligned}$$

**Remark 1.** The form-transformations leave invariant not only the action (2.1) , but also the integrand

$$F = g(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 1) = inv \quad (3.7)$$

Generically there are transformations, which change the signature of the metric. This is possible in particular because the action (2.1) can be rewritten in the form (2.4) , where the metric  $g_{\mu\nu}$  appears *with only lower indices*, namely,

$$S = \int d^2z F^{1/2} \quad (3.8)$$

where

$$F = g_{00}(\partial_1 \phi)^2 - 2g_{01}\partial_0 \phi \partial_1 \phi + g_{11}(\partial_0 \phi)^2 - g_{00}g_{11} + g_{01}^2 \quad (3.9)$$

Hence, there is no singularity in the action in the point  $g = 0$ . Nevertheless, it is useful to find the form-transformations that produce the degenerate metric. For a given field configuration  $\phi(z)$  and  $g_{\mu\nu}(z)$  this degenerating transformations  $\zeta(z)$  are given by the following equation:

$$h_{00}(\partial_0 \zeta - \partial_0 \phi)^2 + h_{11}(\partial_1 \zeta - \partial_1 \phi)^2 - 2h_{01}(\partial_0 \zeta - \partial_0 \phi)(\partial_1 \zeta - \partial_1 \phi) - F = 0 \quad (3.10)$$

where

$$\begin{aligned} h_{00} &= g_{00} - (\partial_0 \phi)^2 \\ h_{11} &= g_{11} - (\partial_1 \phi)^2 \\ h_{01} &= g_{01} - (\partial_0 \phi)(\partial_1 \phi) \end{aligned}$$

It is important to note that any given metric can be transformed (at least locally) via form-transformations to a metric with a definite signature<sup>1</sup> (say, positive) since for the fields belonging to the domain of definition (2.2) the equation (3.10) always define some hyperboloid on a plane  $(\partial_0 \zeta, \partial_1 \zeta)$  which divides the plane into the regions with different signatures of the metric.

**Remark 2.** The action (2.1) is form-invariant also in case when the dimension of the world-sheet is not equal to two.

#### 4. Gauge fixing, resulting free action, tracelessness of the energy-momentum tensor in the gauge

The theory, defined by the action (2.1), is a gauge theory with 3-parametric family of gauge transformations (2 parameters define diffeomorphism and 1 parameter defines form-transformation). Now we can fix a gauge in the classical theory in the following way (imposing 3 gauge condition):

$$g_{00} + g_{11} - (\partial_0 \phi)^2 - (\partial_1 \phi)^2 = 0 \quad (4.1a)$$

$$g_{01} - \partial_0 \phi \partial_1 \phi = 0 \quad (4.1b)$$

$$g_{00} - g_{11} = 0 \quad (4.1c)$$

Let us also note that first two of these gauge conditions (i. e. (4.1a, b)) are invariant with respect to form-transformations and, hence, could be considered as fixing only the

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<sup>1</sup> It is interesting that from (3.10) follows that if we consider not a "time-like action" (i.e. real on time-like surfaces, see Section 5) but a "space-like action"

$$S[g_{\mu\nu}, \phi] = \int d^2 z \sqrt{g} \sqrt{1 - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi} \quad (3.11)$$

which is real on space-like surfaces, than there would be no form-transformation which makes the metric degenerate because in that case the equation (3.10) would have no solution.

reparametrization invariance. Furthermore, in Section 5 below we will see that these two conditions in the framework of a four-dimensional string interpretation of the action (2.1) come from the usual orthonormality gauge conditions in the Nambu-Goto string formalism (see [5]):

$$\begin{aligned}\dot{X}^2 + X'^2 &= 0 \\ (\dot{X}X') &= 0\end{aligned}$$

By analogy with that case, we will call the first two conditions (4.1a, b) "orthonormality conditions".

The third gauge condition (4.1c) is not form-invariant and fixes the form-symmetry. This gauge condition can be always resolved locally.

Now all the components of the metric can be written in terms of  $\phi$ :

$$(g_{\mu\nu}) = \frac{1}{2} \begin{pmatrix} (\partial_0\phi)^2 + (\partial_1\phi)^2 & 2\partial_0\phi\partial_1\phi \\ 2\partial_0\phi\partial_1\phi & (\partial_0\phi)^2 + (\partial_1\phi)^2 \end{pmatrix}$$

and

$$(g^{\mu\nu}) = \frac{2}{((\partial_0\phi)^2 - (\partial_1\phi)^2)^2} \begin{pmatrix} (\partial_0\phi)^2 + (\partial_1\phi)^2 & -2\partial_0\phi\partial_1\phi \\ -2\partial_0\phi\partial_1\phi & (\partial_0\phi)^2 + (\partial_1\phi)^2 \end{pmatrix}$$

Thus, this 3 gauge conditions allow us to exclude completely the metric from the action (in a classical theory). The field  $\phi$  remains the only dynamical variable. Although the possibility of substitution of gauge conditions into the action in general case is a subtle point even on a classical level, we assume that in this case the substitution is valid.

After simple calculations we find that the resulting action is the action of free scalar massless field:

$$S^G[\phi] = \pm \frac{1}{2} \int d^2z ((\partial_0\phi)^2 - (\partial_1\phi)^2) = \int d^2z \sqrt{\det(g^G(\phi))} \quad (4.2)$$

The index  $G$  in  $S^G$  and  $g^G$  denotes that these terms are taken in the gauge (4.1a, b, c). The indeterminacy of the sign of the r. h. s. of (4.2) corresponds to indeterminacy of the choice of zeroth and first coordinate. Let us assume the (+) sign. Than the zeroth coordinate  $z^0$  plays the role of the time coordinate.

The equation of motion for the field  $\phi(z)$  which follows from the action (4.2) is

$$\ddot{\phi} - \phi'' = 0$$

and can also be obtained by substituting of the gauge conditions (4.1a, b, c) into the equation (2.5) .

In the particular gauge (4.1) we find

$$T_{00}^G = -T_{11}^G = \frac{1}{4}((\partial_0\phi)^2 - (\partial_1\phi)^2) \quad (4.3a)$$

$$T_{01}^G = 0 \quad (4.3b)$$

and, hence,

$$T_\mu^{G\mu} = g^{G\mu\nu} T_{\mu\nu}^G = 0 \quad (4.4)$$

i.e. in this gauge the energy-momentum tensor is traceless.

**Remark 1.** The choice of the third gauge condition as  $g_{00} = g_{11}$  (together with orthonormality conditions) is equivalent to the condition  $T_\mu^{G\mu} = 0$  and can be replaced by it. In other words, our gauge is fixed by the orthonormality conditions and by the condition of tracelessness of the resulting energy-momentum tensor.

Let us stress that there is an ambiguity in the choice of the third gauge condition (4.1c) and, moreover, still it is unclear for us should one fix the whole form-symmetry or should one fix only it's subgroup.

Apparently, one should choose the gauge condition in accordance with a topology of the given two-dimensional manifold. The gauge condition (4.1c) can be imposed locally, but for imposing a global gauge one need to perform more sophisticated analysis.

**Remark 2.** One should be cautious about using in this theory the term "gauge symmetry" and "gauge conditions", since in the usual sence they refers to symmetries of the classsical equations of motion and these are absent in the theory. We use the term "gauge symmetry" in a sense that **action** (and not equations of motion) is invariant with respect to transformations.

**Remark 3.** One can try to impose the following gauge condition:  $\phi^\zeta(z) \equiv 0$  thus excluding the field  $\phi(z)$  from the theory. However, it can be argued that it is actually a "bad" choice of a gauge condition for the reasons which are discussed in Section 5.

## 5. Interpretation as 4-dimensional string theory and possible 4-dimensional quantization

The action (functional) (2.1) depending on  $\phi(z)$  and  $g_{\mu\nu}(z)$  which has a positive signature<sup>1</sup> actually equals to the area of some definite 2-dimensional manifold  $\Sigma^2$  imbedded into the 4-dimensional Minkovski space  $R^{3,1}$ . The corresponding imbedding

$$I^{g,\phi} : \Sigma^2 \longrightarrow R^{3,1}$$

is given by  $g_{\mu\nu}(z)$  and  $\phi(z)$  and is built as follows :

**Step 1.** Let us find some **isometric imbedding**  $I_0^g$  of a 2-dimensional surface  $\Sigma^2$  into 3-dimensional Euclidean space  $R^3$ .  $I_0^g$  realizes the metric  $g_{\mu\nu}$  as the metric induced on  $\Sigma^2$  by the imbedding:

$$I_0^* : \eta_{ij}^{(Euclidean)} \longrightarrow g_{\mu\nu}$$

The imbedding  $I_0^g$  *naively* exists and is unique (modulo global shifts and rotations of the resulting surface in Euclidean space ) because the number of the components of metric (three:  $g_{00}(z^1, z^2), g_{01}(z^1, z^2), g_{11}(z^1, z^2)$ ) coincides with the number of functions  $X^i(z^1, z^2)$  ( $i = 1, 2, 3$ ) which parametrically define the imbedding. In order to prove the existence and uniqueness of such imbedding one has to prove the global existence and uniqueness of the solution of the nonlinear nonhomogenous differential equation

$$\partial_\mu X^i \partial_\nu X^i = g_{\mu\nu} \tag{5.1}$$

with respect to  $X^i(z)$ .

The answer to this question strongly depends on the class of imbeddings. As it was shown in [6], if one consider imbeddings of the class  $C^\infty$  (i.e. functions  $X^i(z)$  belong to  $C^\infty$ ) than any compact two-dimensional Riemann manifold can be isometrically embedded into  $R^{10}$ . In other words, in general case one needs **ten** smooth functions  $X^i(z)$  in order to realize an arbitrary metric  $g_{\mu\nu}(z)$  by (5.1) .

However the situation considerably changes for the case of  $C^1$  imbeddings. Nash and Kuiper [7] have shown that if one consider imbeddings of class  $C^1$  than any closed two-dimensional Riemann manifold can be isometrically imbedded into  $R^3$ ! This seems

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<sup>1</sup> Any metric can be transformed via form-transformations to a metric having positive signature, see Section 3.

surprising: for instance, there exists an isometric imbedding of the flat torus into three-dimensional Euclidian space. Evidently, for such imbeddings the metric (5.1) belongs to the class  $C^0$ .

Uniqueness of such an imbedding is not clear, but let us *assume* that this imbedding is unique.

**Step 2.** Let us assign

$$X^0(z) = \phi(z) \quad (5.2)$$

where  $X^0$  is the forth coordinate. Now we have one-to-one (due to the assumption above) correspondence:

$$(g_{\mu\nu}(z), \phi(z)) \Rightarrow (X^0(z), X^1(z), X^2(z), X^3(z)) \quad (5.3)$$

where four functions  $X^\alpha(z)$  ( $\alpha = 0, \dots, 3$ ) belong to  $C^1$  and define parametrically some imbedding of the  $\Sigma^2$  into  $R^{3,1}$ . We will call this imbedding  $I^{g,\phi}$ .

The inverse to (5.3) mapping

$$(g_{\mu\nu}(z), \phi(z)) \Leftarrow (X^0(z), X^1(z), X^2(z), X^3(z)) \quad (5.4)$$

is built simply by equating:  $g_{\mu\nu} = \partial_\mu X^i \partial_\nu X^i$  ( $i = 1, 2, 3$ ) and  $X^0 = \phi$ .

**Preposition.** The action (2.1) gives the area of the 2-dimensional surface  $\Sigma^2$  imbedded into  $R^{1,3}$  by the imbedding  $I^{g,\phi}$ :

$$S[g_{\mu\nu}, \phi] = Area[\Sigma^2]_{I^{g,\phi}} \quad (5.5)$$

*Proof.* This preposition is proved simply by substituting of the (5.1) and (5.2) into the action (2.1). The resulting expression is the area of the surface  $\Sigma^2$  imbedded into  $R^{1,3}$ , i.e.

$$\begin{aligned} S[g_{\mu\nu} \rightarrow \partial_\mu X^i \partial_\nu X^i, \phi \rightarrow X^0] &= \int d^2 z \sqrt{\det(-\eta_{\alpha\beta} \partial_\mu X^\alpha \partial_\nu X^\beta)} \\ &= \int d^2 z \sqrt{\det((\dot{X} X')^2 - \dot{X}^2 X'^2)} \end{aligned}$$

where  $\eta_{\alpha\beta} = \text{diag}(+, -, -, -)$  is a Mincowski metric and  $\alpha, \beta = 0, \dots, 3$ .

*Q.E.D.*

Thus, one may say that the theory (2.1) defines some string theory in 4 dimensions.

The metric induced on the  $\Sigma_2$  by the imbedding into four-dimensional Mincowski space is

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \partial_\mu \phi \partial_\nu \phi \quad (5.6)$$

Let us note that in the gauge (4.1a, b, c)  $\det \tilde{g} = \det g$  (and this condition can replace the third gauge condition (4.1c) fixing the form-invariance)

It is now easy to see that, in fact, first two of the three gauge conditions (4.1a, b, c), in the framework of the 4-dimensional interpretation, are equivalent to the following gauge conditions usually imposed in canonical quantization of the Nambu-Goto string theory:

$$\begin{aligned} \dot{X}^2 + X'^2 &= 0 \\ (\dot{X}X') &= 0 \end{aligned}$$

**Remark 1.** One can write down the point particle action analogous to the action (2.1) :

$$S_P[e(\tau), \phi(\tau)] = \int d\tau \sqrt{e} \sqrt{e^{-1} \dot{\phi}^2 - 1} \quad (5.7)$$

which is reparametrization invariant (  $\tau \rightarrow \tau'(\tau)$  ) and form-invariant (  $\delta_\zeta \phi = \zeta$ ,  $\delta_\zeta e = 2\dot{\phi}\dot{\zeta}$  ). This action has the analogous geometrical meaning: it equals to the length of a corresponding curve in 1 + 1- Mincowski space. This can be seen by substituting into the action (5.7)  $e = \dot{\phi}^2$ . In fact, this point action is useful as a some kind of a toy model for string action (2.1) .

Let us stress the speciality of 4 dimensions for action (2.1) . In principle, we could consider the action (2.1) as the area of the surface imbedded into target Mincowski space which is  $D$ -dimensional ( $D > 4$ ): we would only need to realize the metric  $g_{\mu\nu}$  as induced by an imbedding of the surface into  $R^{D-1}$  ( $D - 1 > 3$ ) and this is certainly possible. But than the correspondence between metric and imbedding definitely would be not one-to-one.

Now having that interpretation of the action (2.1) as the area of the surface imbedded into 4-dimensional Mincowski space one can try to develop some corresponding 4-dimensional string theory. Let us **define** correlational functions of that hypothetic theory as follows:

$$\langle X^\alpha(z_1) \dots X^\gamma(z_n) \rangle_\theta = \int Dg_{\mu\nu}(z) D\phi(z) X_{g,\phi}^\alpha(z_1) \dots X_{g,\phi}^\gamma(z_n) e^{-\theta S} \quad (5.8)$$

where we denote by  $X_{g,\phi}^\alpha(z)$  the solution of equations (5.1) and (5.2) ,i.e. **they are expressed** via  $g_{\mu\nu}(z)$  and  $\phi(z)$ . Integration goes over  $g_{\mu\nu}(z)$  and  $\phi(z)$  and one should

build a proper functional measure and indicate the class of functions over which goes integration. This is, in fact, difficult problem which should be addressed elsewhere and we, by analogy with path integral of quantum mechanics, just assume that integration goes over  $g_{\mu\nu}(z) \in C^0$  and  $\phi(z) \in C^1$ . Than, due to above mentioned results of Nash and Kuiper, in the path integral all imbeddings of Riemann surfaces in four-dimensional Mincowski space are produced without missing any surface and without overcounting any surface.

Let us admit as  $D_{\hat{g}}g_{\mu\nu}D_{\hat{g}}\phi$  the standard formally defined Polyakov measure [3] :  $\int D_{\hat{g}}\delta\phi \exp(-\int \sqrt{\hat{g}}(\delta\phi)^2) = 1$  and  $\int D_{\hat{g}}\delta g_{\mu\nu} \exp(-\frac{1}{2}\int \sqrt{\hat{g}}(2\hat{g}_{\mu\nu}\hat{g}_{\rho\sigma} + \hat{g}_{\mu\rho}\hat{g}_{\nu\sigma})\delta g^{\mu\nu}\delta g^{\rho\sigma}) = 1$

This functional measure is invariant with respect to diffeomorfisms, i.e.  $D_{\hat{g}'}g_{\mu\nu}D_{\hat{g}'}\phi = D_{\hat{g}}g_{\mu\nu}D_{\hat{g}}\phi$  (and, hence,  $Z_{\theta}[g'_{\mu\nu}] = Z_{\theta}[\hat{g}_{\mu\nu}]$ ). One can write equivalently:

$$\langle D_{\mu}T^{\mu\nu} \rangle = 0 \quad (5.9)$$

i.e. the vacuum amplitude of the classical identity (3.1)corresponding to diffeomorfism invariance equals to zero.

Now we should answer the question: is the measure invariant with respect to form-transformations. The possible anomaly of the measure can be obtained via perturbation theory (see for a review [8] ). In this approach the presence of an anomaly is connected with violation of the corresponding symmetry by the regularization procedure. For instance, Weyl invariance of the Polyakov string is violated by the regularization (e.g. dimensional) and, hence, Weyl anomaly arises.

In our case, since the form-symmetry is valid for all dimensions of a world-sheet space (see the Remark 2 in Section 3), dimensional regularization will not violate the form-symmetry and there will be no "form-anomalies" in the theory. The vacuum amplitude of the corresponding identity (3.3) equals to zero (if we use dimensional regularization):

$$\langle \Phi \rangle = \langle T^{\mu\nu} D_{\mu}\partial_{\nu}\phi \rangle \quad (5.10)$$

**Lorentz invariance.** The correlational functions of the four-dimensional theory defined by (5.8) are globally Lorentz covariant. First, let us show the covariance with respect to global Lorentz boosts. In order to show that let us consider two field configurations (i. e. imbeddings)  $X^{\alpha}(z)$  and  $X^{\alpha'}(z)$  connected by an infinitesimal boost:

$$X^{0'} = X^0 + \epsilon X^1$$

$$X^{1'} = X^1 - \epsilon X^0$$

$$X^{2'} = X^2$$

$$X^{3'} = X^3$$

It is easy to see that two field configurations  $(g_{\mu\nu}, \phi)$  and  $(g'_{\mu\nu}, \phi')$  corresponding to  $X^\alpha$  and  $X^{\alpha'}$ , accordingly, (see (5.4) ) are connected by an infinitesimal form transformation with parameter  $\zeta(z) = \epsilon X^1(z)$ :

$$\begin{aligned} g'_{\mu\nu} &= g_{\mu\nu}^\zeta|_{\zeta=\epsilon X^1} \\ \phi' &= \phi^\zeta|_{\zeta=\epsilon X^1} \end{aligned}$$

and, since the action  $S$  is form-invariant and the measure is equal to  $e^{-S}$ , both configurations have in the path integral (5.8) equal weights. Hence, the measure of the path integral is invariant with respect to global Lorentz boosts. Let us also notice that Lorentz transformations are realized on fields  $(g_{\mu\nu}, \phi)$  *nonlocally*.

Since the mappings (5.3) and (5.4) are invariant with respect to global  $SO(3)$  rotations in the space  $X^i$  ( $i = 1, 2, 3$ ), the measure of the functional integral (5.8) is globally  $SO(3)$ -invariant as well. Together with global boost invariance of the measure this yields the global Lorentz covariance of the correlational functions, defined by (5.8) .

Here comes some subtle point. If we are going to consider form-symmetry as a gauge symmetry than we have to define as a physical variables only scalars of a gauge group. Since the Lorentz transformations are subset of the gauge group, one should consider only Lorentz *invariants* as physical variables. But we want Lorentz *covariants* to be included into the set of physical variables as well. There are two way-outs:

Approach 1. Factorize not over the whole group of form-transformations but over form-transformations minus Lorentz transformations. In this case gauge conditions must be Lorentz invariant. Than one can use usual definition of physical variables (i.e. scalars of the gauge group). In this approach one should demonstrate the invariance of the measure of path integral with respect to (non-local) Lorentz transformations. This invariance can depend on the choice of a gauge condition.

Approach 2. Factorize over the whole group of form transformations and consider as physical variables not only the scalars of the gauge group but also the variables which transform **covariantly** under the action of Lorentz transformations and **invariantly** under the action of the other form-transformations. This does not coincide with the standard definition of the physical variables. In this approach the vacuum amplitudes of physical variables *do depend* on the choice of gauge condition in the following way: they transform covariantly under Lorentz transformations of the gauge condition.

In Section 4 (Remark 3) we noted that the gauge condition  $\phi(z) \equiv 0$  is not good. The argument for that comes from Lorentz covariance. This condition evidently is not Lorentz invariant and, hence, cannot be used in the first approach. It is also not good for the second approach since problems with definition of the physical variables arise.

Let us notice that the third gauge conditions (4.1c) in Chapter 3 is non-Lorentz invariant.

## 6. Discussion

The main goal of the consideration of the action (2.1) is a possibility to apply **matrix model technology** for the **four-dimensional string theory**.

It is well known (see [1] ) that matrix models correspond to the case of (initially) **one-dimensional** target space, i. e. when there are only one field  $X(z)$  (and metric  $g_{\mu\nu}(z)$ ) which eventually produces second field: Liouville field  $\varphi(z)$ .

One can demonstrate this as follows (see [1] for details). Consider the partition function

$$Z = \sum_g \int Dg_{\mu\nu}(z) D\phi(z) e^{-S_0} \quad (6.1)$$

where

$$S_0 = \int d^2z \sqrt{g} \left( \frac{1}{\alpha'} g^{\mu\nu} \partial_\mu X \partial_\nu X + \Phi R + \lambda \right) \quad (6.2)$$

Discretizing the action (6.2) (approximating surfaces by collection of equilateral triangles of area  $S_\nabla$ ) we get for path integral (6.1)

$$Z(g_0, \kappa) = \sum_h g_0^{2h-2} \sum_\Lambda \kappa^V \prod_{i=1}^V \int dX_i \prod_{\langle ij \rangle} e^{-(X_i - X_j)^2} \quad (6.3)$$

where  $g_0 = e^\Phi$ ,  $\kappa = e^{-\lambda S_\nabla}$ . First sum runs over genus of discretized surfaces. Second sum runs over all distinct lattices  $\Lambda$  and  $V$  is a number of triangles in a lattice. Second product in (6.3) runs over links of a dual lattice ( $X_i$  live on the vertices of a dual lattice). The term  $e^{-(X_i - X_j)^2}$  arised at the rhs of (6.3) because the discretized version of  $\int d^2z \sqrt{g} g^{\mu\nu} \partial_\mu X \partial_\nu X$  is simply  $\sum_{\langle ij \rangle} (X_i - X_j)^2$ , where the sum runs over all the links of the dual lattice.

As was first noted by Kazakov and Migdal[9], a statistical sum of the form (6.3) is generated in the Feinman graph expansion of the quantum mechanics of a  $N \times N$  hermitian matrix. Consider the Euclidean path integral

$$Z = \int D^{N^2} \Phi(\tau) \exp \left[ -\beta \int_T^T d\tau \text{Tr} \left( \frac{1}{2} \dot{\Phi}^2 + \frac{1}{2\alpha'} \Phi^2 - \frac{1}{6} \Phi^3 \right) \right] \quad (6.4)$$

One obtains the sum over all connected Feynmann graphs  $\Lambda$ :

$$\lim_{T \rightarrow \infty} \ln Z = \sum_h N^{2-2h} \sum_\Lambda \kappa^V \prod_{i=1}^V \int_{-\infty}^{\infty} d\tau_i \prod_{\langle ij \rangle} e^{-|\tau_i - \tau_j|/\alpha'} \quad (6.5)$$

where  $\kappa = \sqrt{N/\beta}$ . The exponential  $e^{-|\tau_i - \tau_j|/\alpha'}$  is the one dimensional massive Euclidean propagator in configuration space. The point is that this expression almost coincides with (6.3) .

The only difference between the two expressions is the exponentials: for the case of two-dimensional gravity we have  $e^{-(X_i - X_j)^2}$  while for the case of matrix model we get  $e^{-|\tau_i - \tau_j|/\alpha'}$ . Usual point of view is that the exact expressions for the exponential are not important especially as matrix model calculations based on partition function (6.4) coincide with those (few) results obtained from continuous calculations in two-dimensional gravity (6.2) .

Let us now consider the action (2.1) . First, let us note that in the action (2.1) the field  $\phi(z)$  must have the dimension of length in order to the expression  $g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$  be dimensionless. If one wish to consider dimensionless field  $\phi(z)$ , one should add some dimensional parameter  $c$  ( $[c] = [l]$ ) into the action:

$$S_c[g_{\mu\nu}, \phi] = \int d^2 z \sqrt{g} \sqrt{c^2 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 1} \quad (6.6)$$

Since  $\phi$  plays the role of fourth coordinate  $X^0$  (or time) in the framework of the four-dimensional interpretation of the action (2.1) (see Section 5, Step 2), we can write:

$$g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \sim 1/v^2$$

where  $v$  is four-dimensional velocity of a point on the propagating string. Hence the action (6.6) is

$$S_c \sim \int d^2 z \sqrt{\frac{c^2}{v^2} - 1}$$

and the physical sense of parameter  $c$  is simply the speed of light.

The case  $\frac{c^2}{v^2} \rightarrow \infty$  corresponds to non-relativistic consideration. In this limit we have

$$S_{NR} = \lim_{c \rightarrow \infty} S_c = c \int d^2 z \sqrt{g} \sqrt{g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi} \quad (6.7)$$

And discretizing this action analogously to (6.2) and considering the partition function we get the expression (6.5) **with correct exponentials**  $e^{-c|\tau_i - \tau_j|}$ , where one can identify  $c = 1/\alpha'$ .

Hence, considering action (6.6) and it's non-relativistic limit one can obtain the **exact** coincidences of the partition functions of 2-dimensional gravity in the form (6.6) and matrix quantum mechanics (6.4) .

**Remark 1.** Considering another limit  $c \rightarrow 0$  (actually  $c^2/v^2 \rightarrow 0$ ), which corresponds to the case when all velocities are much greater than speed of light, we get Polyakov action with one dimensional target space:

$$S_{c \rightarrow 0} = i \left[ \int d^2 z \sqrt{g} - c^2 \int d^2 z \sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right]$$

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### Appendix A. Geometrical meaning of the form-symmetry

Geometrical meaning of the form-symmetry becomes clear in the framework of the four-dimensional interpretation of the action (2.1). This infinite-dimensional symmetry in fact describes the transformation of the shape (or form) of a two-dimensional surface imbedded into four-dimensional target space without change of it's area. For example, one can imagine a two-dimensional sphere and, evidently, it can be **crumpled** having it's area (i.e. action (2.1)) fixed. Intuitively, it is clear that this is indeed infinite-dimensional symmetry.

It is important to note that this symmetry has nothing in common with the reparametrization invariance which maps the image of the imbedding onto itself and, hence, does not change the shape of the imbedded surface. In some sense, the reparametrization invariance is an internal symmetry, while form-symmetry is an external symmetry.

Also, one should distinguish the form symmetry from the area-preserving diffeomorphisms of the target space, because the latter transformations transform a metric of the target space  $G_{\alpha\beta}$  and the form-symmetry in our interpretation does not transform it (the action (2.1) is an area of a surface in Minkowski space). In other words, the form-symmetry does not affect on the target space but changes only an imbedding.

Let us note that the Nambu-Goto action, which is an area of the imbedded surface as well, has also the form-symmetry. But for that case it is not a *local* symmetry, i.e. the form-transformations cannot be written as local transformations of the fields  $X_\alpha(z)$  and, hence, are not gauge transformations. The point is that if we consider  $g_{\mu\nu}$  and  $\phi$  as dynamical variables of the theory (instead  $X_\alpha$ ) the form-symmetry becomes local in new variables and becomes the real gauge symmetry.

## References

- [1] P.Ginsparg and G.Moore, *Preprint YCTP-P23-92* (hep-th/9304011) ;  
I.Klebanov, *Preprint PUPT-1271* (hep-th/9108019)
- [2] A.Carlini and J.Greensite, *Preprint NORDITA-94/71P* (gr-qc/9502023)
- [3] A.M.Polyakov, *Phys.Lett.* **103B** (1981) 207,211
- [4] F.David, *Mod.Phys.Lett.* **A3** (1988) 1651 ;  
J.Distler and H.Kawai, *Nucl.Phys.* **B321** (1989) 509
- [5] J.Scherk, *Rev.Mod.Phys.* **47** (1975) 123
- [6] Gromov M.L. and Rokhlin V.A. *Uspekhi Mat. Nauk* **25** (1970) n.5
- [7] Nash J. *Ann. Math.* **60** (1954) 383-396 ;  
Kuiper N. *I.Proc.Koninkl.nederl.acad.wetensch.* **A 58** (1955) 545-556
- [8] L.Alvarez-Gaumé, in *Unified String Theories*, World Scientific, 1986
- [9] V.Kazakov and A.Migdal, *Nucl.Phys.* **B311** (1989) 171